IIAS Report No. 1997-001, “Infinite Analysis”, pp. 71-80

Orthogonal bases of Calogero-type models

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Abstract

Algebraic structure of Calogero-type models are investigated by using the exchange-operator formalism. It is shown that the set of the Jack polynomials whose arguments are Dunkl-type operators provides an orthogonal basis.

1 Introduction

Among quantum integrable models in one dimension, Calogero-Sutherland type models catch renewed interests, because of not only their physical significance, but also their beautiful mathematical structure. An example of such models is the Calogero model with harmonic potential[1, 2]:

$$H_A = \frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial x_j^2} + x_j^2\right) + \sum_{j<k}\frac{\beta(\beta-1)}{(x_j-x_k)^2}. \tag{1}$$

The subscript *“A”* signifies that this Hamiltonian is invariant under the action of the symmetric group S_N, i.e., the A_{N-1}-type Weyl group. There also exist Calogero-type models associated with other types of the Weyl groups. The B_N-invariant counterpart of the Hamiltonian (1) is the following[3]:

$$\begin{aligned} H_B &= \frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial x_j^2} + x_j^2\right) + \sum_{j=1}^{N}\frac{\gamma(\gamma-1)}{x_j^2} \\ &\quad + \sum_{j<k}\left\{\frac{\beta(\beta-1)}{(x_j-x_k)^2} + \frac{\beta(\beta-1)}{(x_j+x_k)^2}\right\}. \end{aligned} \tag{2}$$

We remark that the model associated with the C_N-type Weyl group is equivalent to the B_N-case, and D_N-type model is obtained by setting $\gamma = 0$.

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Ground state wavefunction for (1) is

$$\psi_0^{(A)}(x) = \prod_{j<k} |x_j - x_k|^\beta \prod_{j=1}^{N} \exp(-x_j^2/2),$$

and that for (2) is

$$\psi_0^{(B)}(x) = \prod_{j<k} |x_j^2 - x_k^2|^\beta \prod_{j=1}^{N} |x_j|^\gamma \prod_{j=1}^{N} \exp(-x_j^2/2).$$

Wavefunctions for excited states can be constructed in principle by using a kind of creation and annihilation operators which include coordinate-exchange[4, 5]. Namely, one can construct the operators with the following commutation relations:

$$[H_W, a_j^{(W)\dagger}] = a_j^{(W)\dagger}, \qquad [H_W, a_j^{(W)}] = -a_j^{(W)}$$

for $W = A$ or B. Using these operators, one can easily see that the wavefunctions of the form,

$$f(a_1^{(W)\dagger}, \ldots, a_N^{(W)\dagger})\psi_0^{(W)}(x) \qquad (W = A \text{ or } B), \tag{3}$$

become eigenstates if $f(x_1, \ldots, x_N)$ are homogeneous polynomials. However naive choice of the polynomials does not create the orthogonal states with respect to the scalar product,

$$(f, g) = \int_{-\infty}^{\infty} f(x_1, \ldots, x_N) g(x_1, \ldots, x_N) \mathrm{d}x.$$

One should properly choose the polynomial $f(x_1, \ldots, x_N)$ of (3).

By using the exchange-operator formalism, we shall show that the proper choice is the Jack symmetric polynomials for both the A_{N-1}-case[6] and the B_N-case[7].

2 Preliminaries

2.1 Creation and annihilation operators

We first introduce *"gauge-transformed"* Hamiltonians:

$$\begin{aligned} \widetilde{H}_A &= (\phi_0^{(A)})^{-1} \circ H_A \circ \phi_0^{(A)} \\ &= \frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial x_j^2} + x_j^2\right) - \frac{\beta}{2}\sum_{j\neq k}\frac{1}{x_j - x_k}\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k}\right), \end{aligned}$$

$$
\begin{aligned}
\widetilde{H}_B &= (\phi_0^{(B)})^{-1} \circ H_B \circ \phi_0^{(B)} \\
&= \frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial x_j^2} + x_j^2 - \frac{2\gamma}{x_j}\frac{\partial}{\partial x_j}\right) - \beta\sum_{k\neq j}\frac{1}{x_j^2 - x_k^2}\left(x_j\frac{\partial}{\partial x_j} - x_k\frac{\partial}{\partial x_k}\right),
\end{aligned}
$$

where $\phi_0^{(A)}$ and $\phi_0^{(B)}$ are polynomial parts of the ground state wavefunctions:

$$
\begin{aligned}
\phi_0^{(A)}(x_1,\ldots,x_N) &= \prod_{j<k}|x_j - x_k|^{\beta}, \\
\phi_0^{(B)}(x_1,\ldots,x_N) &= \prod_{j<k}|x_j^2 - x_k^2|^{\beta}\prod_{j=1}^{N}|x_j|^{\gamma}.
\end{aligned}
$$

We denote transformed ground state as $\widetilde{\psi}_0^{(\mathrm{c})}$, i.e., $\widetilde{\psi}_0^{(\mathrm{c})} = \prod_{j=1}^{N}\exp(-x_j^2/2)$. We remark that this $\widetilde{\psi}_0^{(\mathrm{c})}$ is joint eigenfunction of $\widetilde{H}_A$ and $\widetilde{H}_B$.

Creation and annihilation operators for these Hamiltonians can be constructed in quite similar way as in the case of the harmonic oscillators[3, 5]:

$$
A_j^{(W)\dagger} = \frac{1}{\sqrt{2}}\left(-D_j^{(W)} + x_j\right), \qquad A_j^{(W)} = \frac{1}{\sqrt{2}}\left(D_j^{(W)} + x_j\right)
$$

for $W = A$ or B. Here we have used so-called *"Dunkl operators"*[8]:

A_{N-1}**-type**

$$
D_j^{(A)} = \frac{\partial}{\partial x_j} + \beta\sum_{k(\neq j)}\frac{1}{x_j - x_k}(1 - s_{jk}) \qquad (j = 1,\ldots,N),
$$

B_N**-type**

$$
\begin{aligned}
D_j^{(B)} &= \frac{\partial}{\partial x_j} + \gamma\sum_{j=1}^{N}\frac{1}{x_j}(1 - t_j) \\
&\quad + \beta\sum_{k(\neq j)}\left\{\frac{1}{x_j - x_k}(1 - s_{jk}) + \frac{1}{x_j + x_k}(1 - t_j t_k s_{jk})\right\},
\end{aligned}
$$

where the operators s_{jk} and t_j act as follows:

$$
s_{ij}f(\ldots,x_i,\ldots,x_j,\ldots) = f(\ldots,x_j,\ldots,x_i,\ldots),
$$

$$
t_j f(\ldots,x_j,\ldots) = f(\ldots,-x_j,\ldots).
$$

Commutation relations for these operators are the following:

A_{N-1}**-type**

$$
\begin{aligned}
[A_i^{(A)\dagger}, A_j^{(A)\dagger}] &= [A_i^{(A)}, A_j^{(A)}] = 0, \\
[A_i^{(A)}, A_j^{(A)\dagger}] &= \delta_{ij}\left(1 + \beta \sum_{k(\neq i)} s_{ik}\right) - (1-\delta_{ij})\beta s_{ij}, \\
s_{ij}A_j^{(A)\dagger} &= A_i^{(A)\dagger}s_{ij}, \quad s_{ij}A_j^{(A)} = A_i^{(A)}s_{ij}, \\
s_{ij}A_k^{(A)\dagger} &= A_k^{(A)\dagger}s_{ij}, \quad s_{ij}A_k^{(A)} = A_k^{(A)}s_{ij} \quad (k \neq i,j),
\end{aligned}
$$

B_N**-type**

$$
\begin{aligned}
[A_i^{(B)\dagger}, A_j^{(B)\dagger}] &= [A_i^{(B)}, A_j^{(B)}] = 0, \\
[A_i^{(B)}, A_j^{(B)\dagger}] &= \delta_{ij}\left(1 + \beta \sum_{k(\neq i)} (s_{ik} + t_i t_k s_{ik})\right) \\
&\quad - (1-\delta_{ij})\beta(s_{ij} - t_i t_k s_{ik}). \\
s_{ij}A_j^{(B)} = A_i^{(B)}s_{ij}, &\qquad s_{ij}A_k^{(B)} = A_k^{(B)}s_{ij} \quad (k \neq i,j), \\
t_j A_j^{(B)} = -A_i^{(B)}t_j, &\qquad t_j A_k^{(B)} = A_k^{(B)}t_j \quad (k \neq j).
\end{aligned}
$$

If we define $\widehat{H}_W$ as

$$
\widehat{H}_W = \frac{1}{2}\sum_{j=1}^{N}\left(A_j^{(W)}A_j^{(W)\dagger} + A_j^{(W)\dagger}A_j^{(W)}\right) \qquad (W = A \text{ or } B),
$$

then $\widehat{H}_W$ are related to the gauge-transformed Calogero Hamiltonians:

$$
\widetilde{H}_W = \mathrm{Res}\left(\widehat{H}_W\right),
$$

where $\mathrm{Res}(X)$ means that action of an operator X is restricted to symmetric functions of the variables $x_1, \ldots, x_N$ for A_{N-1}-case, or to symmetric functions of the variables $x_1^2, \ldots, x_N^2$ for B_N-case. Hence, if we consider wavefunctions with above symmetry, spectrum of $\widetilde{H}_W$ coincides with that of $\widehat{H}_W$. Hereafter we will consider $\widehat{H}_W$ instead of $\widetilde{H}_W$.

Although the commutation relations for $A_j^{(W)}$ and $A_j^{(W)\dagger}$ are complicated, commutation relations with $\widehat{H}_W$ are quite simple:

$$
[\widehat{H}_W, A_j^{(W)\dagger}] = A_j^{(W)\dagger}, \qquad [\widehat{H}_W, A_j^{(W)}] = -A_j^{(W)}.
$$

Using the operators $A_j^{(W)\dagger}$, one can construct excited state wavefunctions:

$$f\left(A_1^{(A)\dagger},\dots,A_N^{(A)\dagger}\right)\widetilde{\psi}_0^{(\mathrm{c})} \qquad \text{for the } A_{N-1} \text{ case,}$$
$$f\left((A_1^{(B)\dagger})^2,\dots,(A_N^{(B)\dagger})^2\right)\widetilde{\psi}_0^{(\mathrm{c})} \qquad \text{for the } B_N \text{ case,}$$

with $f(x_1,\dots,x_N)$ symmetric polynomials with homogeneous degree.

2.2 Symmetric and non-symmetric Jack polynomials

First we define *"Cherednik operators"* $\widehat{D}_j^{(A)}$ [9, 10]:

$$\begin{aligned}
\widehat{D}_j^{(A)} &= x_j D_j^{(A)} + \beta \sum_{k(<j)} s_{jk} \\
&= x_j \frac{\partial}{\partial x_j} - \beta \sum_{k(<j)} \frac{x_k}{x_j - x_k}(s_{jk}-1) \\
&\qquad -\beta \sum_{k(>j)} \frac{x_j}{x_j - x_k}(s_{jk}-1) + \beta(j-1).
\end{aligned} \tag{4}$$

Since the operators $\widehat{D}_j^{(A)}$ commute each other, they are diagonalized simultaneously by suitable choice of bases of $\mathbb{C}[x_1,\dots,x_N]$ [10, 11]. Such basis is called *non-symmetric* Jack polynomials. An non-symmetric Jack polynomial $E_w^\lambda(x)$, labeled with the partition $\lambda=(\lambda_1,\dots,\lambda_N)$ and the element $w\in S_N$, is characterized by the following properties[10, 11]:

1. $E_w^\lambda(x) = x_w^\lambda + \sum_{(\mu,w')<(\lambda,w)} C_{ww'}^{\lambda\mu} x_{w'}^\mu$,

2. $E_w^\lambda(x)$ is joint eigenfunctions for the operators $\widehat{D}_j^{(A)}$,

where we have used the notation $x_w^\lambda = x_{w(1)}^{\lambda_1}\cdots x_{w(N)}^{\lambda_N}$. To define the ordering $(\mu,w')<(\lambda,w)$, we use the dominance ordering $<_{\mathrm{D}}$ for partitions[12], and the Bruhat ordering $<_{\mathrm{B}}$ for the elements of S_N. Using these, we define the ordering as follows:

$$(\mu,w')<(\lambda,w) \quad \Longleftrightarrow \quad \begin{cases} \text{(i)} & \mu <_{\mathrm{D}} \lambda, \\ \text{(ii)} & \text{if } \mu=\lambda \text{ then } w' <_{\mathrm{B}} w. \end{cases}$$

We denote the eigenvalues of $\widehat{D}_j^{(A)}$ as $\epsilon_j(\lambda,w)$:

$$\widehat{D}_j^{(A)} E_w^\lambda(x) = \epsilon_j(\lambda,w) E_w^\lambda(x).$$

The eigenvalues $\epsilon_j(\lambda,w)$ are all obtained by permutating the components of the multiplet $\{\lambda_{N-j+1}+\beta(j-1)\}_{j=1,\dots,N}$.

Using the operator $\widehat{D}_j^{(A)}$, we introduce generating function of symmetric commuting operators[10]:

$$\widehat{\Delta}_{\mathrm{S}}^{(A)}(u) = \prod_{j=1}^{N}(u + \widehat{D}_j^{(A)}).$$

Since $\widehat{\Delta}_{\mathrm{S}}^{(A)}(u)$ is symmetric in $\widehat{D}_j^{(A)}$, symmetric eigenfunctions are obtained by symmetrizing $E_w^{\lambda}(x)$, which are nothing but the Jack symmetric polynomials $J_\lambda(x)$. Eigenvalues of $\widehat{\Delta}_{\mathrm{S}}^{(A)}(u)$ are then given by

$$\widehat{\Delta}_{\mathrm{S}}^{(A)}(u)J_\lambda(x) = \prod_{j=1}^{N}\left\{u + \lambda_{N-j+1} + \beta(j-1)\right\} J_\lambda(x). \tag{5}$$

We note that all the eigenvalues of $\widehat{\Delta}_{\mathrm{S}}^{(A)}(u)$ are distinct for generic values of u.

3 Construction of orthogonal bases

3.1 A_{N-1}-type model

We denote the algebra generated by the elements x_j, $D_j^{(A)}$ and s_{ij} as $\mathcal{A}_{\mathrm{S}}^{(A)}$. We then introduce an $\mathcal{A}_{\mathrm{S}}^{(A)}$-module $\mathcal{F}_{\mathrm{S}}^{(A)}$ (*"Fock space"*) generated by the vacuum vector $\widetilde{\psi}_0^{(\mathrm{s})} = 1$. The elements $D_j^{(A)}$ of $\mathcal{A}_{\mathrm{S}}^{(A)}$ annihilate the vacuum vector, and s_{ij} preserve $\widetilde{\psi}_0^{(\mathrm{s})}$, i.e.,

$$D_j^{(A)}\widetilde{\psi}_0^{(\mathrm{s})} = 0, \qquad s_{ij}\widetilde{\psi}_0^{(\mathrm{s})} = \widetilde{\psi}_0^{(\mathrm{s})}.$$

We denote an algebra generated by $A_j^{(A)}$, $A_j^{(A)\dagger}$ and s_{ij} as $\mathcal{A}_{\mathrm{C}}^{(A)}$. Since the commutation relations of these operators are the same as those of x_j and $D_j^{(A)}$, we can introduce an isomorphism of $\mathcal{A}_{\mathrm{S}}^{(A)}$ to $\mathcal{A}_{\mathrm{C}}^{(A)}$ as follows:

$$\rho(x_j) = A_j^{(A)\dagger}, \qquad \rho(D_j^{(A)}) = A_j^{(A)}.$$

We then extend ρ to the isomorphism of Fock spaces. Fock space for $\mathcal{A}_{\mathrm{C}}^{(A)}$ is constructed in the same way as $\mathcal{A}_{\mathrm{S}}^{(A)}$; Fock space $\mathcal{F}_{\mathrm{C}}^{(A)}$ is defined as $\mathcal{F}_{\mathrm{C}}^{(A)} = \mathbb{C}[A_1^{(A)\dagger},\ldots,A_N^{(A)\dagger}]\widetilde{\psi}_0^{(\mathrm{c})}$ where the vacuum vector $\widetilde{\psi}_0^{(\mathrm{c})} = \prod_{j=1}^{N}\exp(-x_j^2/2)$ is annihilated by $A_j^{(A)}$, i.e., $A_j^{(A)}\widetilde{\psi}_0^{(\mathrm{c})} = 0$. We denote also by ρ the isomorphism of $\mathcal{F}_{\mathrm{S}}^{(A)}$ to $\mathcal{F}_{\mathrm{C}}^{(A)}$ such that

$$\rho(\widetilde{\psi}_0^{(\mathrm{s})}) \;=\; \widetilde{\psi}_0^{(\mathrm{c})} = \prod_{j=1}^{N}\exp(-x_j^2/2),$$

$$\rho(a\widetilde{\psi}_0^{(\mathrm{s})}) \quad = \quad \rho(a)\widetilde{\psi}_0^{(\mathrm{c})} \in \mathbb{C}[A_1^{(A)\dagger},\ldots,A_N^{(A)\dagger}]\widetilde{\psi}_0^{(\mathrm{c})}$$

for $a \in \mathbb{C}[x_1,\ldots,x_N]$.

Since the operators $\widehat{D}_j^{(A)}$ commute each other, we can construct commuting operators $\widehat{h}_j^{(A)}$ acting on $\mathcal{F}_{\mathrm{C}}^{(A)}$ as

$$\widehat{h}_j^{(A)} = \rho(\widehat{D}_j^{(A)}) = A_j^{(A)\dagger}A_j^{(A)} + \beta\sum_{k(<j)} s_{jk}.$$

Generating function of symmetric commuting operators that include $\widehat{H}_A$ is constructed by using $\widehat{h}_j^{(A)}$:

$$\widehat{\Delta}_{\mathrm{C}}^{(A)}(u) = \rho\left(\widehat{\Delta}_{\mathrm{S}}^{(A)}(u)\right) = \prod_{j=1}^{N}(u+\widehat{h}_j^{(A)}).$$

Applying ρ to (5), we have the following eigenvalue equation for $\widehat{\Delta}_{\mathrm{C}}^{(A)}(u)$:

$$\begin{aligned}
&\widehat{\Delta}_{\mathrm{C}}^{(A)}(u)J_\lambda\left(A_1^{(A)\dagger},\ldots,A_N^{(A)\dagger}\right)\widetilde{\psi}_0^{(\mathrm{c})}\\
&\quad = \prod_{j=1}^{N}\{u+\lambda_{N-j+1}+\beta(j-1)\}\,J_\lambda\left(A_1^{(A)\dagger},\ldots,A_N^{(A)\dagger}\right)\widetilde{\psi}_0^{(\mathrm{c})}.
\end{aligned}$$

Since all the eigenvalues of $\widehat{\Delta}_{\mathrm{C}}^{(A)}(u)$ are distinct and the operator $\widehat{h}_j^{(A)}$ is self-adjoint with respect to the scalar product,

$$(f,g)_{\mathrm{C}}^{(A)} = \int_{-\infty}^{\infty} f(x_1,\ldots,x_N)g(x_1,\ldots,x_N)(\phi_0^{(A)})^2\mathrm{d}x,$$

we now know that the wavefunctions $J_\lambda(A^{(A)\dagger})\widetilde{\psi}_0^{(\mathrm{c})}$ form an orthogonal basis for the A_{N-1} case.

3.2 B_N-type model

Using the B_N-type Dunkl operators, we can define another set of commuting operator $\widehat{D}_j^{(B)}$:

$$\begin{aligned}
\widehat{D}_j^{(B)} &= z_jD_j^{(B)} + \beta\sum_{k(<j)}(s_{jk}+t_jt_ks_{ik})\\
&= z_j\frac{\partial}{\partial z_j} + \beta\sum_{k(<j)}\left\{\frac{z_k}{z_j-z_k}(1-s_{jk}) - \frac{z_k}{z_j+z_k}(1-t_jt_ks_{jk})\right\}\\
&\quad +\beta\sum_{k(>j)}\left\{\frac{z_j}{z_j-z_k}(1-s_{jk}) + \frac{z_j}{z_j+z_k}(1-t_jt_ks_{jk})\right\} + 2\beta(j-1),
\end{aligned}$$

where we have replaced the variables x_j with z_j for the latter convenience. If we restrict the action of $\widehat{D}_j^{(B)}$ to the functions with the symmetry $t_j f(z) = f(z)$, then $\widehat{D}_j^{(B)}$ is reduced to

$$\begin{aligned}\mathrm{Res}^{(t)}(\widehat{D}_j^{(B)}) &= z_j \frac{\partial}{\partial z_j} + 2\beta \sum_{k(<j)} \frac{z_k^2}{z_j^2 - z_k^2}(1 - s_{jk}) \\ &\quad + 2\beta \sum_{k(>j)} \frac{z_j^2}{z_j^2 - z_k^2}(1 - s_{jk}) + 2\beta(j-1). \qquad (6)\end{aligned}$$

Comparing (6) with (4), we find that $\mathrm{Res}^{(t)}(\widehat{D}_j^{(B)})$ is equivalent to $2\widehat{D}_j^{(A)}$ if we make a change of the variables $z_j = x_j^2/2$.

Defining the operator $\widehat{\Delta}_{\mathrm{S}}^{(B)}(u)$ as

$$\widehat{\Delta}_{\mathrm{S}}^{(B)}(u) = \prod_{j=1}^{N} (u + \widehat{D}_j^{(B)}),$$

we have the following equation by using the correspondence between $\mathrm{Res}^{(t)}(\widehat{D}_j^{(B)})$ and $2\widehat{D}_j^{(A)}$:

$$\begin{aligned}&\widehat{\Delta}_{\mathrm{S}}^{(B)}(u) J_\lambda\left(x_1^2/2, \ldots, x_N^2/2\right) \\ &\quad = \prod_{j=1}^{N} \{u + 2\lambda_{N-j+1} + 2\beta(j-1)\} J_\lambda\left(x_1^2/2, \ldots, x_N^2/2\right). \qquad (7)\end{aligned}$$

We call the algebra generated by the elements x_j, $D_j^{(B)}$, s_{ij} and t_j as $\mathcal{A}_{\mathrm{S}}^{(B)}$, and the algebra generated by $A_j^{(B)}$, $A_j^{(B)\dagger}$, s_{ij} and t_j as $\mathcal{A}_{\mathrm{C}}^{(B)}$. Since the commutation relations of these operators are the same as those of x_j and $D_j^{(B)}$, we can define an isomorphism σ of $\mathcal{A}_{\mathrm{S}}^{(B)}$ to $\mathcal{A}_{\mathrm{C}}^{(B)}$ as follows:

$$\sigma(x_j) = A_j^{(B)\dagger}, \qquad \sigma(D_j^{(B)}) = A_j^{(B)}.$$

We then introduce Fock spaces for $\mathcal{A}_{\mathrm{S}}^{(B)}$ and $\mathcal{A}_{\mathrm{C}}^{(B)}$:

$$\begin{aligned}\mathcal{F}_{\mathrm{S}}^{(B)} &= \mathbb{C}[x_1^2, \ldots, x_N^2]\widetilde{\psi}_0^{(\mathrm{s})}, \\ \mathcal{F}_{\mathrm{C}}^{(B)} &= \mathbb{C}[(A_1^{(B)\dagger})^2, \ldots, (A_N^{(B)\dagger})^2]\widetilde{\psi}_0^{(\mathrm{c})}.\end{aligned}$$

The elements $D_j^{(B)}$ of $\mathcal{A}_{\mathrm{S}}^{(B)}$ annihilate $\widetilde{\psi}_0^{(\mathrm{s})}$, while $A_j^{(B)\dagger}$ annihilate $\widetilde{\psi}_0^{(\mathrm{c})}$. Hence the isomorpism σ can be extended to the isomorphism of the Fock spaces:

$$\sigma(\widetilde{\psi}_0^{(\mathrm{s})}) = \widetilde{\psi}_0^{(\mathrm{c})}, \qquad \sigma(a\widetilde{\psi}_0^{(\mathrm{s})}) = \sigma(a)\widetilde{\psi}_0^{(\mathrm{c})}$$

for $a \in \mathbb{C}[x_1^2, \ldots, x_N^2]$.

Applying this isomorphism to (7), we have the following eigenvalue equation for $\widehat{\Delta}_{\mathrm{C}}^{(A)}(u)$:

$$\widehat{\Delta}_{\mathrm{C}}^{(B)}(u) J_\lambda \left((A_1^{(B)\dagger})^2/2, \ldots, (A_N^{(B)\dagger})^2/2 \right) \widetilde{\psi}_0^{(\mathrm{c})}$$
$$= \prod_{j=1}^{N} \{u + 2\lambda_{N-j+1} + 2\beta(j-1)\} J_\lambda \left((A_1^{(B)\dagger})^2/2, \ldots, (A_N^{(B)\dagger})^2/2 \right) \widetilde{\psi}_0^{(\mathrm{c})}.$$

Since all the eigenvalues of $\widehat{\Delta}_{\mathrm{C}}^{(B)}(u)$ are distinct and the operator $\widehat{h}_j^{(B)}$ is self-adjoint with respect to the scalar product,

$$(f, g)_{\mathrm{C}}^{(B)} = \int_{-\infty}^{\infty} f(x_1, \ldots, x_N) g(x_1, \ldots, x_N) (\phi_0^{(B)})^2 \mathrm{d}x,$$

we conclude that the wavefunctions $J_\lambda \left((A^{(B)\dagger})^2/2 \right) \widetilde{\psi}_0^{(\mathrm{c})}$ form an orthogonal basis for the B_N case.

4 Concluding remarks

We have constructed an orthogonal basis for the Calogero-type models by using the Jack polynomials whose arguments are Dunkl-type operators:

A_{N-1}**-type model:** $\left\{ J_\lambda \left(A_1^{(A)\dagger}, \ldots, A_N^{(A)\dagger} \right) \widetilde{\psi}_0^{(\mathrm{c})} \right\} \phi_0^{(A)}$,

B_N**-type model:** $\left\{ J_\lambda \left((A_1^{(B)\dagger})^2/2, \ldots, (A_N^{(B)\dagger})^2/2 \right) \widetilde{\psi}_0^{(\mathrm{c})} \right\} \phi_0^{(B)}$.

In both cases, wavefunctions are of the form,

(symmetric polynomials) $\times$ (ground state wavefunction).

The polynomial parts may be regarded as a multi-variable generalization of classical orthogonal polynomials[13, 14]. In case of the A_{N-1}-type model, they are a multi-variable generalization of the Hermite polynomials while they are a multi-variable generalization of the Laguerre polynomials in the B_N case.

It should be noted that the norms of these orthogonal wavefunctions have been calculated via some limiting procedure[13, 14]. However dynamical correlation functions have not been calculated so far, due to the lack of some formulas needed. We hope that our results provide a useful tool for deeper understanding of the Calogero-type models.

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